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On scalar curvature in lightlike geometry

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Abstract

We introduce the concept of induced scalar curvature of a class $\mathcal{C}[M]$ of lightlike hypersurfaces $(M, g, S(TM))$, of a Lorentzian manifold, such that M admits a canonical screen distribution $S(TM)$, a canonical lightlike transversal vector bundle and an induced symmetric Ricci tensor. We prove that there exists such a class $\mathcal{C}[M]$ of a globally hyperbolic warped product spacetime [J.K. Beem, P.E. Ehrlich, K.L. Easley, Global Lorentzian Geometry, 2nd edition, Marcel Dekker, Inc. New York, 1996, MR1384756 (97f:53100)] of general relativity. In particular, we calculate the scalar curvature of a member of $\mathcal{C}[M]$ in a globally hyperbolic spacetime of constant curvature, supported by an example.

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1. Introduction

Let M be a hypersurface of an $(m + 2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index $0 < \nu < m + 2$. In the theory of nondegenerate hypersurfaces, the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of main geometric objects. In the case of lightlike hypersurfaces, the situation is different. The normal bundle $TM^{\perp}$ is a rank-one distribution over M: $TM^{\perp} \subset TM$ and coincides with the so called radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$. Hence, the induced metric g on M is degenerate and has constant rank m. A complementary bundle of Rad TM in TM is a rank m nondegenerate distribution over M, called a *screen distribution* on M, denoted by $S(TM)$. The existence of $S(TM)$ is secured provided M be paracompact.

In this paper, we study lightlike hypersurfaces with a specific screen distribution, denoted by $(M, g, S(TM))$, of a Lorentzian manifold $(\bar{M}, \bar{g})$. Let F be a timelike sub bundle of $S(TM)^{\perp}$. It is known [6] that for any null section ξ of $\mathrm{Rad}(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of the lightlike transversal bundle $ltr(TM)$ satisfying

$$\bar{g}(N, \xi) = 1, \qquad \bar{g}(N, N) = \bar{g}(N, W) = 0, \quad \forall\, W \in \Gamma(ST(M)|_{\mathcal{U}}) \tag{1}$$

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if and only if N is given by

$$N = \frac{1}{\bar{g}(\xi, V)} \left\{ V - \frac{g(V, V)}{2\, \bar{g}(\xi, V)}\, \xi \right\}, \tag{2}$$

where $V \in \Gamma(F_{|\mathcal{U}})$ is a timelike vector such that $\bar{g}(\xi, V) \neq 0$. Then TM is decomposed as follows:

$$TM = \mathrm{Rad}(TM) \oplus S(TM).$$

We denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle E over M, $\mathcal{F}(M)$ being the algebra of smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.

In general, the induced linear connection, say ∇, on M is not a Levi-Civita connection and depends on both g and a screen distribution $S(TM)$ of M. Let $\bar{\nabla}$ be the Levi-Civita connection on $(\bar{M}, \bar{g})$ and P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. Consider a normalizing pair $\{\xi, N\}$ satisfying (1). Then, the local Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{3}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \tau(X) = \bar{g}(\nabla^t_X N, \xi), \tag{4}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi,$$

$$\nabla_X \xi = -A^*_\xi - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{5}$$

where B and C are the local second fundamental forms on $\Gamma(TM)$ and $\Gamma(S(TM))$, respectively, ∇^* is a metric connection on $\Gamma(S(TM))$, A^*_ξ the local shape operator on $S(TM)$ and τ a 1-form on TM. It is well known that the second fundamental form and the shape operator of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, in the case of lightlike hypersurfaces, there are interrelations between these geometric objects and those of its screen distributions. More precisely, it is easy to see that the two local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{6}$$

$$C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N Y, N) = 0. \tag{7}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\, TM$ [9]. As per [6, page 83], the second fundamental form B of M is independent of the choice of a screen distribution and satisfies

$$B(X, \xi) = 0, \quad \forall X, Y \in \Gamma(TM).$$

It is important to mention that there are large classes of lightlike hypersurfaces with canonical and or integrable screen distribution [6,7]. Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively. The induced tensor of type (0, 2) on M is given by

$$R^{(0,2)}(X, Y) = \mathrm{trace}\{Z \to R(X, Z)Y\}, \quad \forall\, X, Y \in \Gamma(TM). \tag{8}$$

Since the induced connection ∇ on M is not a Levi-Civita connection, in general, $R^{(0,2)}$ is not symmetric. Therefore, in general, it is just a tensor quantity and has no geometric or physical meaning similar to the symmetric Ricci tensor of $\bar{M}$. Observe that there are a variety of lightlike hypersurfaces which admit an induced symmetric Ricci tensor (see, for example, [6,7]).

As the induced metric g on M is degenerate, its inverse does not exist. Thus, it is not possible to find, from the Eq. (8) of $R^{(0,2)}$ tensor, a scalar quantity for M, by the usual way of contraction. This raises the following question. Is there any other way to associate M with a scalar quantity which we may call an *induced scalar curvature* of M? Intuitively, the answer must be "YES", since M being a smooth paracompact manifold, several important concepts, related to the scalar curvature of the ambient manifold $\bar{M}$, can have no induced corresponding objects on M if the answer is "NO". Moreover, the semi-Riemannian geometry of an integral manifold of an integrable screen distribution remains incomplete if the answer is "NO".

The objective of this paper is to propose a way to heal this missing gap by introducing the concept of induced scalar curvature for a class of lightlike hypersurfaces of Lorentzian manifolds.

2. Lightlike hypersurfaces of genus zero

To introduce a concept of induced scalar curvature for a lightlike hypersurface M we observe that, in general, the non-uniqueness of screen distribution $S(TM)$ and its non-degenerate causal structure rules out the possibility of a definition for an arbitrary M of a semi-Riemannian manifold. Although, as mentioned in the introduction, there are many cases of a canonical screen (some of them integrable) and canonical transversal vector bundle, the problem of scalar curvature must be classified subject to the causal structure of a screen. For this reason, in this paper we start with a hypersurface $(M, g, S(TM))$ of a Lorentzian manifold $(\bar{M}, \bar{g})$ for which we know that any choice of $S(TM)$ is Riemannian. This case is also physically important (see [1,4,5,7,8] and many more cited therein). Using the relation (see [6, page 93, equation (3.1)]) of $\bar{R}$ and R and the Gauss–Weingarten equations we obtain the following Gauss–Codazzi equations of M:

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \tag{9}$$

$$\bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + B(X, Z)C(Y, W) - B(Y, Z)C(X, W)$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)\xi, N) &= g(R(X, Y)\xi, N) \\ &= C(Y, A_\xi X) - C(X, A_\xi Y) - 2d\tau(X, Y). \end{aligned} \tag{10}$$

Let $\{\xi; W_a\}$ be the quasi-orthonormal frame for TM induced from a frame $\{\xi; W_a, N\}$ for $T\bar{M}$ such that $S(TM) = \mathrm{Span}\,\{W_1, \ldots, W_m\}$ and $\mathrm{Rad}\,TM = \mathrm{Span}\,\{\xi\}$. Setting $X = Y$ in (8) provides the following local expression of $R^{(0,2)}(X, X)$ in terms of the above frame (see details in [6, page 98]):

$$R^{(0,2)}(X, X) = \sum_{b=1}^{m} g(R(X, W_b)X, W_b) + \bar{g}(R(X, \xi)X, N). \tag{11}$$

Replacing X by ξ and using the above Gauss–Codazzi equations we obtain

$$\begin{aligned} R^{(0,2)}(\xi, \xi) &= \sum_{a=1}^{m} g(R(\xi, W_a)\xi, W_a) - \bar{g}(R(\xi, \xi)\xi, N) \\ &= \sum_{a=1}^{m} g(R(\xi, W_a)\xi, W_a), \end{aligned} \tag{12}$$

where the second term vanishes due to (10). Replacing each X successively by each base vector W_a of $S(TM)$ and then taking the sum we get

$$\sum_{a=1}^{m} R^{(0,2)}(W_a, W_a) = \sum_{a=1}^{m} \left\{ \sum_{b=1}^{m} g(R(W_a, W_b)W_a, W_b) \right\} + \sum_{a=1}^{m} \bar{g}(R(W_a, \xi)W_a, N). \tag{13}$$

Finally, adding (12) and (13) we obtain a scalar r given by

$$\begin{aligned} r &= R^{(0,2)}(\xi, \xi) + \sum_{a=1}^{m} R^{(0,2)}(W_a, W_a) \\ &= \sum_{a=1}^{m} \left\{ \sum_{b=1}^{m} g(R(W_a, W_b)W_a, W_b) \right\} + \sum_{a=1}^{m} \left\{ g(R(\xi, W_a)\xi, W_a) + \bar{g}(R(W_a, \xi)W_a, N) \right\}. \end{aligned} \tag{14}$$

In general, r given by (14) can not be called a scalar curvature of M since it is not possible to calculate it from a tensor quantity $R^{(0,2)}$. Moreover, it can only have a geometric meaning if $R^{(0,2)}$ is symmetric and its value is independent of the screen, its transversal vector bundle and the null section ξ. We, therefore, need reasonable geometric conditions on M to recover its scalar curvature. For this purpose we introduce the following general concept.

We say that a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ and any induced object of it, denoted by $\Omega^{\mathbf{s}}$, are of *genus* $\mathbf{s}$ if the induced metric tensor $g_{|S(TM)}$ is of constant signature $\mathbf{s}$.

Although the above general concept is introduced for possible study of a scalar curvature of lightlike hypersurfaces of a prescribed semi-Riemannian manifold, in this paper, we restrict its use to a M in a Lorentzian manifold $\bar{M}$.

Then, we say that M (labeled by $M^{\mathbf{0}}$) is a lightlike hypersurface of genus zero with screen $S(TM)^{\mathbf{0}}$. Denote by $\mathcal{C}[M^{\mathbf{0}}] = [(M^{\mathbf{0}}, g^{\mathbf{0}}, S(TM)^{\mathbf{0}})]$ a class of lightlike hypersurfaces of genus zero such that

(a) $M^{\mathbf{0}}$ admits a canonical screen distribution $S(TM)^{\mathbf{0}}$ that induces a canonical lightlike transversal vector bundle $N^{\mathbf{0}}$
(b) $M^{\mathbf{0}}$ admits an induced symmetric Ricci tensor, denoted by $\mathrm{Ric}^{\mathbf{0}}$.

For geometric and/or physical reasons, the above two conditions are necessary to assign a well-defined scalar curvature to each member of $\mathcal{C}[M^{\mathbf{0}}]$.

3. Induced scalar curvature of genus zero

For the convenience of readers we recall the following three local transformation equations (see [6, page 87, equation (2.26) and (2.27)]) of a change in $S(TM)$ to another screen $S(TM)'$:

$$W'_a = \sum_{b=1}^{m} W_a^b\,(W_b - \mathbf{f}_b\,\xi)\,, \tag{15}$$

$$N' = N - \frac{1}{2}\left\{\sum_{a=1}^{m}(\mathbf{f}_a)^2\right\}\xi + \sum_{a=1}^{m}\mathbf{f}_a W_a. \tag{16}$$

$$\nabla'_X Y = \nabla_X Y + B(X,Y)\left\{\frac{1}{2}\left(\sum_{a=1}^{m}(\mathbf{f}_a)^2\right)\xi - \sum_{a=1}^{m}\mathbf{f}_a\,W_a\right\}, \tag{17}$$

where $\{W_a\}$ and $\{W'_a\}$ are the local orthonormal basis of $S(TM)$ and $S'(TM)$ with respective transversal sections N and N' for the same null section ξ. Denote by $\mathcal{S}$ the first derivative of a screen $S(TM)$ given by

$$\mathcal{S}(x) = \mathrm{span}\{[X,Y]_x,\ X_x, Y_x \in S(TM),\ x \in M\}. \tag{18}$$

where [,] denotes the Lie-bracket. If $S(TM)$ is integrable, then, $\mathcal{S}$ is a sub-bundle of $S(TM)$.

Definition 1. Let $(M^{\mathbf{0}}, g^{\mathbf{0}}, S(TM)^{\mathbf{0}}, \xi^{\mathbf{0}}, N^{\mathbf{0}})$ be a member of $\mathcal{C}[M^{\mathbf{0}}]$. Then, the scalar quantity r, given by (14) and denoted by $r^{\mathbf{0}}$, is called the induced scalar curvature of genus zero of $M^{\mathbf{0}}$.

Since $S(TM)^{\mathbf{0}}$ and $N^{\mathbf{0}}$ are chosen canonical, we must assure the stability of $r^{\mathbf{0}}$ with respect to a choice of the second fundamental form $B^{\mathbf{0}}$ and the 1-form $\tau^{\mathbf{0}}$. Choose another null section $\xi' = \alpha\xi^{\mathbf{0}}$. Then, with respect to ξ' there is another $N' = (1/\alpha)N^{\mathbf{0}}$ satisfying (1). Using the Gauss–Weingarten Eqs. (3) and (4) we obtain

$$B' = \alpha B^{\mathbf{0}} \quad \text{and} \quad \tau(X) = \tau'(X) + X(\log\alpha), \quad \forall X \in \Gamma(TM_{\mathcal{U}}).$$

It is easy to see that for a canonical vector bundle $N^{\mathbf{0}}$, the function α in the above relation will be a non-zero constant which implies that, for this case, both $B^{\mathbf{0}}$ and $\tau^{\mathbf{0}}$ are independent of the choice of $\xi^{\mathbf{0}}$, except for a non-zero constant factor. Consequently, $r^{\mathbf{0}}$ is a well-defined induced scalar curvature of a class of lightlike hypersurfaces of genus zero.

Now we show that there exists a variety of Lorentzian manifolds which admit lightlike hypersurfaces of class $\mathcal{C}[M^{\mathbf{0}}]$. First we ask the following question: Is the local screen second fundamental form C independent of the choice of a screen distribution $S(TM)$? The answer is negative. Indeed, we prove the following with respect to a change in $S(TM)$.

Proposition 1. *The screen second fundamental forms C and C' of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows:*

$$C'(X,PY) = C(X,PY) - \frac{1}{2}\|W\|^2 B(X,Y) + g(\nabla_X PY, W), \tag{19}$$

where $W = \sum_{a=1}^{m}\mathbf{f}_a\,W_a$ is the characteristic vector field of the screen change.

Proof. Using (16) and (17) we get

$$\begin{aligned} C'(X, PY) &= \bar{g}(\nabla'_X PY, N') \\ &= \bar{g}\left(\nabla_X PY + B(X,Y)\left\{\frac{1}{2}\left(\sum_{a=1}^{m}(\mathbf{f}_a)^2\right)\xi - \sum_{a=1}^{m}\mathbf{f}_a W_a\right\}, N'\right) \\ &= \bar{g}(\nabla_X PY, N) + \bar{g}\left(\nabla_X PY, \sum_{a=1}^{m}\mathbf{f}_a W_a\right) \\ &\quad + B(X,Y)\left\{\frac{1}{2}\left(\sum_{a=1}^{m}(\mathbf{f}_a)^2\right) - \sum_{b=1}^{m}\sum_{a=1}^{m} g(\mathbf{f}_a W_a, \mathbf{f}_b W_b)\right\} \\ &= C(X, PY) + g(\nabla_X PY, W) - \frac{1}{2}\|W\|^2 B(X,Y) \end{aligned}$$

which is the desired formula. □

Denote by ω the dual 1-form of $W = \sum_{a=1}^{m} \mathbf{f}_a\, W_a$ with respect to the induced metric g of M, that is,

$$\omega(X) = g(X, W), \quad \forall X \in \Gamma(TM). \tag{20}$$

Mathematical model. Consider the following warped product spacetime manifold $(\bar{M}, \bar{g})$ (including globally hyperbolic spacetimes [3]) given by

$$\bar{M} = L \times_f F, \qquad \bar{g} = -\mathrm{e}^{\lambda}\mathrm{d}t^2 + \mathrm{e}^{\mu}(\mathrm{d}x^1)^2 \oplus f\, g' \tag{21}$$

with respect to a local coordinate system $(t, x^1, \ldots, x^{m+1})$, where λ and μ are functions of t and x^1 alone, L is a two-dimensional spacetime, (F, g') its complete Riemannian submanifold of codimension 2, f is a strictly positive warped function on L. Let $\mathbf{E} = \{e_0, e_1, \ldots, e_{m+1}\}$ be an orthonormal basis, such that e_0 is timelike and all others are spacelike unit vectors. Transform $\mathbf{E}$ into another basis $\mathcal{E} = \{\partial_u, \partial_v, e_2, \ldots, e_{m+1}\}$ such that ∂_u and ∂_v are real null vectors satisfying $\bar{g}(\partial_u, \partial_v) = 1$ with respect to a new coordinate system $\{u, v, x^2, \ldots, x^{m+1}\}$. Then, the line element of $\bar{g}$ transforms into

$$\mathrm{d}s^2 = -A^2(u,v)\mathrm{d}u\,\mathrm{d}v + f\, g'_{ab}\mathrm{d}x^a\,\mathrm{d}x^b, \quad 2 \le a, b \le m+1,$$

for some function $A(u, v)$ on M. The absence of $\mathrm{d}u^2$ and $\mathrm{d}v^2$ in the above line element implies that $\{v = \text{constant}\}$ and $\{u = \text{constant}\}$ are lightlike hypersurfaces and their intersection provides a leaf of their common screen distribution whose Riemannian metric is given by $\mathrm{d}\Omega^2 = f\, g'_{ab}\mathrm{d}x^a\mathrm{d}x^b$. Let $(M, g, S(TM), v = \text{constant})$ be a lightlike hypersurface of $\bar{M}$. Let us call $\mathcal{G} = \{\lambda, \mu, f, g'_{ab}\}$ the generating set for a family of lightlike hypersurfaces for prescribed values of its elements. In particular, we denote by $\mathcal{G} = \{0, \mu, f, g'_{ab}\}$ the generating set of all globally hyperbolic warped product spacetimes [3] of general relativity.

Theorem 1. *Let $(M, g, S(TM), v = \text{constant})$ be a lightlike hypersurface of a globally hyperbolic warped product spacetime $(\bar{M}, \bar{g}, \mathcal{G})$, with $\bar{g}$ given by (21) for a prescription $\mathcal{G} = (0, \mu, f, g'_{ab})$. Then,*

(a) *there exists a section ξ of* Rad *TM with respect to which M admits an integrable screen distribution $S(TM)$.*
(b) *The 1-form ω in (20) vanishes identically on the first derivative $\mathcal{S}$ given by (18).*
(c) *If $\mathcal{S}$ coincides with $S(TM)$, then, M admits a canonical screen distribution, up to an orthogonal transformation and a canonical lightlike transversal vector bundle. The screen second fundamental form C is independent of the choice of a screen distribution. Moreover, if the 1-form τ induced by the screen distribution is closed (i.e., $\mathrm{d}\tau = 0$) on any $\mathcal{U} \in M$, then, M belongs to the class $\mathcal{C}[M]$.*

Proof. Let $(t, x^1, \ldots, x^{m+1})$ be local coordinates on $\bar{M}$. Assume that Rad *TM* is spanned by a null vector field ξ. Let $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m+1}}\}$ be a natural basis with respect to which the null vector field ξ is given by

$$\xi = \xi^t \frac{\partial}{\partial t} + \sum_{\alpha=1}^{m+1} \xi^\alpha \frac{\partial}{\partial x^\alpha}, \qquad \sum_{\alpha=1}^{m+1} (\xi^\alpha)^2 = (\xi^t)^2. \tag{22}$$

It is well-known that $\bar{M}$, given by (21) with the prescription $\mathcal{G} = (0, \mu, f, g'_{ab})$, has at least one timelike covariant constant vector field. Thus we may choose (without any loss of generality), along M, a timelike covariant constant vector field $V = -\frac{\partial}{\partial t} \in \Gamma(F_{|\mathcal{U}})$ which satisfies the condition $\bar{g}(V, \xi) = \xi^t \neq 0$ of (2). Then, the vector bundle $\mathcal{B} = \mathrm{Span}\{\frac{\partial}{\partial t}, \xi\}$ is non-degenerate on M. Take the complementary orthogonal vector bundle $S(TM)$ to $\mathcal{B}$ in $T\bar{M}$, which is a non-degenerate distribution on M complementary to Rad TM. This means that $S(TM)$ is a screen distribution on M such that $\mathcal{B} = S(TM)^{\perp}$, which we choose. Using this, (2) and (22) we obtain

$$N = (\xi^t)^{-1}\left(V + \frac{1}{2\,\xi^t}\,\xi\right), \tag{23}$$

the null transversal vector bundle of M. Using (23) in (4) and (5) we get

$$\begin{aligned}\tau(X) &= \bar{g}(\bar{\nabla}_X N, \xi) = X(\xi^t)^{-1}\,\bar{g}(V, \xi) + \frac{1}{2}\,(\xi^t)^{-2}\,\bar{g}(\bar{\nabla}_X\,\xi, \xi)\\ &= X(\xi^t)^{-1}\,\xi^t = X(\log\,\theta),\end{aligned} \tag{24}$$

where we set $(\xi^t)^{-1} = \theta$. Using this value of τ, (23) and (5) we get

$$\begin{aligned}\bar{\nabla}_X\,N &= X(\theta)V + \theta X(\theta)\xi + \frac{1}{2}\,\theta^2\,\bar{\nabla}_X\,\xi\\ &= X(\theta)V + \frac{1}{2}\theta X(\theta)\xi - \frac{1}{2}\,\theta^2\,A^*_{\xi}X.\end{aligned} \tag{25}$$

On the other hand, substituting the value of τ in (4), we get

$$\bar{\nabla}_X\,N = -A_N\,X + X(\theta)V + \frac{1}{2}\theta\,X(\theta)\,\xi. \tag{26}$$

Equating (25) with (26) and putting back $\theta = (\xi^t)^{-1}$ we get

$$A_N = \frac{1}{2}(\xi^t)^{-2}\,A^*_{\xi}. \tag{27}$$

It follows from (6) that, for any lightlike hypersurface, the shape operator A^*_{ξ} of $S(TM)$ is symmetric with respect to g. This result and (27) means that, for a particular class of hypersurfaces of Theorem 1, the shape operator A_N of M is symmetric with respect to g which further implies from [6, page 89] that our chosen screen distribution $S(TM)$ is integrable. This proves (a).

To prove (b), using (6), (7) and (26) we obtain

$$\begin{aligned}C(X, PY) &= g(A_N X, PY) = \frac{1}{2}(\xi^t)^{-2}g(A^*_{\xi}X, PY)\\ &= \frac{1}{2}(\xi^t)^{-2}B(X, PY) = \frac{1}{2}(\xi^t)^{-2}B(X, Y), \quad \forall X, Y \in \Gamma(TM_{|\mathcal{U}}).\end{aligned}$$

The right hand side of the above relation being symmetric in X and Y we get

$$g(\nabla_X PY - \nabla_Y PX, W) = 0, \quad \forall\,X, Y \in \Gamma(TM).$$

Thus, we have $g(\nabla_X Y - \nabla_Y X, W) = 0$ for any $X, Y \in \Gamma(S(TM))$, that is, $\omega([X, Y]) = g([X, Y], W) = 0$ for any $X, Y \in \Gamma(S(TM))$, which proves (b).

Now assume that $\mathcal{S} = S(TM)$, that is, ω vanishes on $S(TM)$, which implies from (20) that $W = 0$ and, therefore, the functions $\mathbf{f}_a$ vanish. Thus, the transformation Eqs. (15) and (16) becomes $W'_a = \sum_{b=1}^{m} W_a^b W_b$ $(1 \leq a \leq m)$ and $N' = N$ where (W_a^b) is an orthogonal matrix of $S(T_x M)$ at any $x \in M$, which proves the first part of (c). The independence of C follows by putting $W = 0$ in (19). Finally, if $\mathrm{d}\tau = 0$ on any $\mathcal{U} \in M$, then, it follows from Theorem 3.2 of [6, page 99] that the induced Ricci tensor of M is symmetric. This, together with all the above shows that M belongs to the class $\mathcal{C}[M]$ which completes the proof of Theorem 1. □

Physical model. A spacetime with the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + S^2(t)\mathrm{d}\Sigma^2$$

is called a *Robertson–Walker spacetime* where $\mathrm{d}\Sigma^2$ is the metric of a spacelike hypersurface Σ with spherical symmetry and constant curvature $c = 1, -1$ or 0. With respect to a local spherical coordinate system (r, θ, ϕ), this metric is given by

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + h^2(r)\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2),$$

where the function $h(r) = \sin r$, $\sinh r$ or r according as $c = 1, -1$ or 0. The range of the coordinates is restricted from 0 to 2π or from 0 to ∞ for $c = 1$ or -1 respectively.

It is known [3] that all Robertson–Walker spacetimes are globally hyperbolic. Following as in the Mathematical model one can show that Robertson–Walker spacetimes admit lightlike hypersurfaces, generated from the family of set $\mathcal{G}$ for prescribed values of its elements. Proceeding as in the proof of Theorem 1, one can show (left as an exercise) that there exists a class $\mathcal{C}[M^{\mathbf{0}}]$ of lightlike hypersurfaces of Robertson–Walker spacetimes.

Observe that the Minkowski space, de-Sitter space and anti-de-Sitter space are all special cases of the general Robertson–Walker spacetimes. For details on these spacetimes and some more (which may satisfy Theorem 1) we refer [3] or any other standard book on general relativity.

4. Example

Consider a lightlike hypersurface $(M^{\mathbf{0}}, g^{\mathbf{0}}, S(TM)^{\mathbf{0}}, \xi^{\mathbf{0}}, N^{\mathbf{0}})$ of a time orientable spacetime manifold $\bar{M}(c)$ of constant curvature c, satisfying Theorem 1. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by [2]

$$\bar{R}(X, Y)Z = c\,\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\}. \tag{28}$$

Using (28) in (10) we have $\bar{g}(R(X, \xi)Y, N) = -c\bar{g}(X, Y)$. Using this result and (28) in (9) reduces the Eq. (11) to

$$\begin{aligned}\mathrm{Ric}^{\mathbf{0}}(X, X) &= \sum_{b=1}^{m} g^{\mathbf{0}}(R(X, W_b)X, W_b) - c\bar{g}(X, X)\\ &= \sum_{b=1}^{m}\Big\{B^{\mathbf{0}}(W_b, X)C^{\mathbf{0}}(X, W_b) - B^{\mathbf{0}}(X, X)C^{\mathbf{0}}(W_b, W_b)\Big\} - cm\bar{g}(X, X).\end{aligned} \tag{29}$$

Consequently, $\mathrm{Ric}^{\mathbf{0}}(\xi^{\mathbf{0}}, \xi^{\mathbf{0}}) = 0$ and the Eqs. (12) and (13) reduce to

$$\begin{aligned}\sum_{a=1}^{m}\mathrm{Ric}^{\mathbf{0}}(W_a, W_a) &= \sum_{a=1}^{m}\left\{\sum_{b1}^{m}[B^{\mathbf{0}}(W_b, W_a)C^{\mathbf{0}}(W_a, W_b) - B^{\mathbf{0}}(W_a, W_b)C^{\mathbf{0}}(W_b, W_b)]\right\} - cm^2\\ &= r^{\mathbf{0}}.\end{aligned} \tag{30}$$

It is immediate from above that, for this class $\mathcal{C}[M^{\mathbf{0}}]$, the induced Ricci tensor and the corresponding scalar curvature of $M^{\mathbf{0}}$ can be determined if one knows $(\mathrm{Ric}^{\mathbf{0}})_{|S(TM)^{\mathbf{0}}}$ and $(r^{\mathbf{0}})_{|S(TM)^{\mathbf{0}}}$. More precisely,

$$\mathrm{Ric}^{\mathbf{0}} = (\mathrm{Ric}^{\mathbf{0}})_{|S(TM)^{\mathbf{0}}}, \quad r^{\mathbf{0}} = (r^{\mathbf{0}})_{|S(TM)^{\mathbf{0}}} - cm^2. \tag{31}$$

We recall from the proof of Theorem 1 that the following holds:

$$C^{\mathbf{0}}(X, PW) = \frac{1}{2}\theta^2 B^{\mathbf{0}}(X, W), \quad \theta = (\xi^t)^{-1}, \forall X, W \in \Gamma(TM^{\mathbf{0}}),$$

where ξ^t is the first coefficient of $\xi^{\mathbf{0}}$ in (22). Taking into account the above relations we obtain the following value of $r^{\mathbf{0}}$ from (30)

$$r^{\mathbf{0}} = \frac{1}{2}\theta^2\sum_{a=1}^{m}\left\{\sum_{b=1}^{m}[(B^{\mathbf{0}})^2(W_a, W_b) - B^{\mathbf{0}}(W_a, W_a)B^{\mathbf{0}}(W_b, W_b)]\right\} - cm^2. \tag{32}$$

For three and four dimensional $\bar{M}(c)$ the values of $r^{\mathbf{0}}$, respectively, are:

$$r^{\mathbf{0}} = -c \quad \text{and} \quad r^{\mathbf{0}} = \theta^2[(B^{\mathbf{0}})^2(W_1, W_2) - B^{\mathbf{0}}(W_1, W_1)B^{\mathbf{0}}(W_2, W_2)] - 4c.$$

Example. Let $(\mathbf{R}^4_1, \bar{g})$ be the Minkowski space with signature $(-,+,+,+)$ of the canonical basis $(\partial_t, \partial_1, \partial_2, \partial_3)$. It follows from [6, pages 115–117] that there exists a lightlike hypersurface of $(\mathbf{R}^4_1, \bar{g})$ with a canonical integrable screen distribution and a canonical lightlike transversal vector bundle $ltr(TM)$. We make this choice for a hypersurface $(M, g, S(TM), \xi, N)$ given by an open subset of the lightlike cone

$$\left\{t(1, \cos u \cos v, \cos u \sin v, \sin u) \in \mathbf{R}^4_1 : t > 0, u \in (0, \pi/2), v \in [0, 2\pi]\right\}.$$
$$\mathrm{Rad}(TM) = \mathrm{Span}\{\xi = \partial_t + \cos u \cos v\, \partial_1 + \cos u \sin v\, \partial_2 + \sin u\, \partial_3\},$$
$$ltr(TM) = \mathrm{Span}\left\{N = \frac{1}{2}(-\partial_t + \cos u \cos v\, \partial_1 + \cos u \sin v\, \partial_2 + \sin u\, \partial_3)\right\},$$

respectively and the screen distribution $S(TM)$ is spanned by

$$\{W_1 = -\sin u \cos v\, \partial_1 - \sin u \sin v\, \partial_2 + \cos u\, \partial_3,\ W_2 = -\sin v\, \partial_1 + \cos v\, \partial_2\}.$$

With respect to a local frame field of $\bar{M}$, the metric $\bar{g}$ has the matrix

$$[\bar{g}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2\cos^2 u \end{bmatrix}.$$

We refer [6, pages 90–99] for details on the general local expressions of the calculations and the notations used. The Christoffel symbols Γ^k_{ij} and the coefficients B_{ij} and A^i_j of the second fundamental form and its associated shape operator respectively, $i, j, k \in \{0, 1, 2\}$, are given by

$$\Gamma^1_{22} = \sin u \cos u, \qquad \Gamma^2_{12} = -\tan u, \qquad \Gamma^1_{1o} = \frac{1}{t}, \qquad \Gamma^2_{2o} = \frac{1}{t};$$
$$\Gamma^o_{11} = -\frac{t}{2}, \qquad \Gamma^o_{22} = -\frac{1}{2} t \cos^2 u, \qquad B_{11} = -t, \qquad B_{22} = -t\cos^2 u,$$
$$A^1_1 = -\frac{1}{2t}, \qquad A^2_2 = -\frac{1}{2t}, \quad \text{and all others vanish.}$$

Using the local Gauss–Codazzi equations and $\bar{R}_{abcd} = 0$, we have

$$\begin{aligned} R_{ac} &= \frac{2}{t}\,\Gamma^0_{ac} + \frac{1}{t^2}\,\Gamma^0_{a1} B_{1c} + \frac{1}{t^2\cos^2 u}\,\Gamma^0_{a2} B_{2c} \\ &= \frac{1}{t}\,B_{ac} + \frac{1}{t^2}\,B_{a1}\Gamma^0_{1c} + \frac{1}{t^2\cos^2 u}\,B_{a2}\Gamma^0_{2c}. \end{aligned}$$

Thus $R_{11} = -\frac{1}{2}$, $R_{12} = R_{21} = 0$ and $R_{22} = -\frac{1}{2}\cos^2 u$. Similarly, we have

$$R_{ao} = \frac{2}{t}\,\Gamma^0_{a0} = 0, \qquad R_{oa} = R_{oo} = 0.$$

Therefore, the Ricci tensor of this lightlike hypersurface is symmetric. Thus, this class of lightlike hypersurfaces belongs to $\mathcal{C}[M]$. The components of the second fundamental form are

$$B(W_1, W_2) = -\left(\frac{1}{t\cos u}\right), \qquad B(W_2, W_2) = -\left(\frac{1}{t\cos u}\right), \qquad B(W_1, W_2) = 0.$$

Since in this example $\xi^t = 1$ (the first coefficient of ξ) it follows from (24) that $\theta = 1$. Using all this data in (32) we conclude that the induced scalar curvature of M in $(\mathbf{R}^4_1, \bar{g})$ is given by

$$r^{\mathbf{0}} = -\frac{1}{(t\cos u)^2}.$$

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References

[1] M.A. Akivis, V.V. Goldberg, The geometry of lightlike hypersurfaces of the de sitter space, Acta Appl. Math. 53 (1998) 297–328. MR 1653456 (2000c:53086).

[2] M. Barros, A. Romero, Indefinite Kaehler manifolds, Math. Ann. 261 (1982) 55–62. MR 0675207 (84d: 53033).

[3] J.K. Beem, P.E. Ehrlich, K.L. Easley, Global Lorentzian Geometry, 2nd edition, Marcel Dekker, Inc., New York, 1996. MR1384756 (97f:53100).

[4] W.B. Bonnor, Null hypersurfaces in Minkowski spacetime, Tensor (N.S.) 20 (1972) 329–345. MR 0334047 (48: 12366).

[5] K.L. Duggal, Constant scalar curvature and warped product globally null manifolds, J. Geom. Phys. 43 (4) (2002) 327–340. MR 1929910 (2004b:53122).

[6] K.L. Duggal, A. Bejancu, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, vol. 364, Kluwer Academic Publishers, Dordrecht, 1996. MR1383318 (97e:53121).

[7] K.L. Duggal, A. Giménez, Lightlike hypersurfaces of Lorentzian manifolds with distinguished screen, J. Geom. Phys. 55 (2005) 107–122. MR 2157417 (2006 b: 53072).

[8] G.J. Galloway, Maximum principles for null hypersurfaces and null splitting theorems, Ann. Henri Poincaré 1 (2000) 543–567. MR 1777311 (2002b:53052).

[9] D.N. Kupeli, Singular Semi-Riemannian Geometry, vol. 366, Kluwer Acad. Publishers, Dordrecht, 1996. MR 1392222 (97f: 53105).